ELASTOPLASTIC BEHAVIOR OF A REINFORCED LAYER

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The mechanical properties of a reinforced material depend on the properties of the binder, the materials of the reinforcing elements, their percentage contents, and nature of the reinforcement. Therefore, in practice each class of reinforced material requires special analysis. Here the primary objective is to obtain the relations describing the connections between the stresses and strains during deformation of the reinforced material.

There are two approaches to the construction of these connections: the phenomenological approach, in which the reinforced medium is considered a homogeneous monolithic anisotropic medium [1, 2], and the approach based on structural analysis of the reinforced material in accordance with the nature of the material structure and the mechanical properties of its constituent components [3, 4]. Within the elastic limits, if we neglect some subtle effects (stress concentration in the vicinity of the reinforcing elements, nonuniformity of the deformations between these elements, and so on), both approaches yield essentially the same coupling equations between the average stresses and strains and in this sense are equivalent.

The phenomenological approach to the formulation of the equations governing the elastoplastic behavior of anisotropic media has been used in several studies, for example [5-7]. However, the equations obtained should apparently be considered applicable to "physically anisotropic media" whose anisotropy is a consequence of their crystalline structure. As for the structurally anisotropic media, including the reinforced materials, beyond the elastic limit the specific characteristics of each type of structural anisotropy affect the form of the plasticity conditions [8-10] and also affect the nature of the coupling equations between the stresses and strains. Therefore, each type of structural anisotropy must be analyzed separately. Therefore, beyond the elastic limit only a structural analysis of the reinforced material on the basis of a model reflecting its specific characteristics permits obtaining the sought coupling relations between the stresses and strains. Another advantage of structural analysis is that it permits evaluating the nature of the operation of each of the elements of the composite and thereby opens up a way to goal-directed regulation of the nature of the reinforcement in order to improve the strength properties of the reinforced materials.

In the following we use the model of [10] and some additional simplifying assumptions to analyze the elastoplastic behavior of a reinforced layer subject to forces in its plane.

1. By a reinforced layer we mean a comparatively thin plate consisting of an isotropic layer with a reinforcing layer embedded in it (Fig. 1). The embedded layer is a grid of slender one-dimensional filaments arranged in directions which form the angles $\alpha_n (n=1, 2, \ldots, N)$ with the direction 1.

We assume:

1) the material of all the elements comprising the composite is elastoplastic and in the general case is different for each element;

2) the number of reinforcing elements is sufficiently large that the material of the composite can be considered quasihomogeneous;

3) the distance between the reinforcing elements is sufficiently large and at the same time sufficiently small in comparison with the plate dimensions that local effects near the filaments and irregularity of the deformation between filaments can be neglected;

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4) the bonding of the elements of the composite is ideal, i.e., there is no slippage between the reinforcing elements and the binder;

5) each filament is capable of withstanding both tensile and compressive forces when embedded in the binder material.

However, under the action of a compressive force some form of instability may arise and therefore the yield and ultimate limits (as well as the strain-hardening modulus for strain-hardening materials) in tension and compression are considered different. The Young's moduli are assumed to be the same in tension and compression;

6) the material of the isotropic binder obeys the deformation theory of plasticity with the same characteristics in tension and compression and for simplicity is assumed to be incompressible in both the elastic and plastic regions. When necessary, the incompressible requirement can be disregarded and we can use a flow type theory rather than deformation theory.

Let ω_n (n = 1, 2, ..., N) be the specific concentrations in the plane of the layer of the reinforcing filaments forming the angles α_n with the direction 1; h is the thickness of the reinforced layer; and ω_z is the reinforcing layer concentration in the direction of the plate thickness. Then in the 1~2 orthogonal system the components of the internal forces in the composite layer will be

$$t_k = a\sigma_{ij}^{\circ} + \sum_{n=1}^{N} \omega_n \sigma_n l_{in} l_{jn} \qquad (i, j = 1, 2;\ k = 1, 2, 3)$$
$$t_j = \frac{T_{jj}}{h}, \quad t_3 = \frac{T_{12}}{h}, \quad l_{1n} = \cos\alpha_n, \quad l_{2n} = \sin\alpha_n \tag{1.1}$$
$$0 \leqslant \alpha_n \leqslant \pi, \ a = 1 - \omega_z, \ \omega_n = n_n F_n / AFh, \ \omega_z = \delta / h$$

Here T_{ij} are the forces, σ_{ij}^0 are the stresses in the filler, σ_n are the stresses in the reinforcing filaments, F_n are the cross-section areas of the reinforcing elements, and n_n is the number of reinforcing element filaments on segments AF of length l (Fig. 2).

On the basis of the assumption of no slippage, for small deformations we obtain the following relations between the deformations ε_n of the reinforcing elements and the deformations of the filler layer:

$$\varepsilon_{(n)} = \varepsilon_1 l_{1n}^2 + \varepsilon_2 l_{2n}^2 + \varepsilon_3 l_{1n} l_{2n} \tag{1.2}$$

Here ε_1, ε_2 are the filler layer deformation components in directions 1 and 2, respectively, and ε_3 is the shear deformation.

In accordance with the assumptions adopted above, the internal stresses of the composite layer elements are connected with the deformations by the following relations:

$$\sigma_{11}^0 = {}^4/_3\ E_c(\varepsilon_1 + {}^1/_2\varepsilon_2), \quad \sigma_{22}^0 = {}^4/_3\ E_c(\varepsilon_2 + {}^1/_2\,\varepsilon_1)$$
$$\sigma_{12}^0 = {}^1/_3\ E_c\varepsilon_3, \quad \sigma_n = E^{\pm}_{cn}\varepsilon_n \tag{1.3}$$

Here E_c, $E_{cn}^{\pm}$ are the secant moduli of the filler and reinforcing element materials in tension (plus) and compression (minus).

If all elements of the composite remain elastic for the given loads t_k (k = 1, 2, 3), then all the secant moduli equal the corresponding Young's moduli

$$E_c = E, \quad E_{cn}^{\pm} = E_n \tag{1.4}$$

Then, substituting (1.3) into (1.1), we obtain the following relationships between the forces t_k and deformations ε_k:

$$\mathbf{t} = \|a_{km}\|\,\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \|b_{km}\|\,\mathbf{t}, \qquad \|b_{km}\| = \|a_{km}\|^{-1}$$
$$\mathbf{t} = \|t_1, t_2, t_3\|', \qquad \boldsymbol{\varepsilon} = \|\varepsilon_1, \varepsilon_2, \varepsilon_3\|' \qquad (k, m = 1, 2, 3) \tag{1.5}$$

The prime on the matrix indicates the transposition operation.

The coefficients of the matrix $\|a_{km}\|$ are

$$a_{ii} = {}^4/_3 aE + \sum_{n=1}^{N} \omega_n E_n l_{in}^4, \quad a_{12} = a_{21} = {}^2/_3 Ea + \sum_{n=1}^{N} \omega_n E_n l_{1n}^2 l_{2n}^2,$$

$$a_{i3} = a_{3i} = \sum_{n=1}^{N} \omega_n E_n l_{in}^3 l_{jn}, \quad a_{33} = {}^1/_3 aE + \sum_{n=1}^{N} \omega_n E_n l_{1n}^2 l_{2n}^2, \qquad (i, j = 1, 2; i \neq j) \tag{1.6}$$

Substituting the deformation components from (1.5) into (1.3) and taking (1.4) into account, we find the internal stresses in all the elements composing the reinforced layer. With their aid we can find the loads for which any particular elements change from the composite into the plastic state. Thus, the reinforcing elements remain elastic if the following inequalities are satisfied:

$$-\sigma_n^- < E_n\Big[\sum_{k=1}^{3} t_k (b_{1k} l_{1n}^2 + b_{2k} l_{2n}^2 + b_{3k} l_{1n} l_{2n})\Big] < \sigma_n^+, \quad (n = 1, 2, \ldots, N) \tag{1.7}$$

and the binder layer remains elastic if the following inequality is satisfied:

$$\sigma_{11}^{02} - \sigma^0_{11}\sigma^0_{22} + \sigma_{22}^{02} + 3\sigma_{12}^{02} < \sigma_0^2 \tag{1.8}$$

where we substitute into this equality in place of σ_{ij}^0 the expressions (1.3) with account for (1.4) and (1.5).

In (1.7), (1.8), the stresses σ_0, $\sigma_n^\pm$ denote respectively the yield limits of the binding and reinforcing element materials in tension or compression.

Violation of any of the inequalities (1.7), (1.8) leads to the development of plastic deformations in the corresponding reinforcing elements or filler. Assume, for example, that for some combination of forces t_k such that

$$f_n(t_1, t_2, t_3) = 0 \tag{1.9}$$

any of the inequalities (1.7) is violated. Then we can assume that (1.9) in the t_1, t_2, t_3 stress space defines the yield surface for the reinforced material with composite elements having elastoplastic properties. In fact, there are no residual deformations for all stresses within this surface after unloading of the layer. Residual deformations remain for stresses outside this surface in the reinforced layer after removal of the load. We also obtain surfaces which are analogous in meaning in stress space in cases in which the other inequalities (1.7) or (1.8) are violated (separately or together). Thus, the combined yield surface for the reinforced material in stress space consists of a large number of "pieces" of different analytic surfaces and its shape depends in an essential fashion on the nature of the reinforcement and the properties of the composite elements.

Moreover, the form of the coupling equations for the elastoplastic behavior of the reinforced layer also depends significantly on the nature of the reinforcement. For example, suppose the plasticity condition has the form (1.9) and the strain diagram of the reinforcing elements has a linear hardening segment. Then we have for stresses located near but outside the surface (1.9)

$$E_c = E, \quad E_{cp}^{\pm} = E_p \quad (p = 1, 2, \ldots, N, \; p \neq n) \tag{1.10}$$

and the strain law in the elements of the angled reinforcement with angle α_n has the form

$$\sigma_n = E_{tn}^{\pm}\varepsilon_n \pm \sigma_n^{\pm}(1 - E_{tn}^{\pm} / E_n) \tag{1.11}$$

Here the upper or lower signs are taken depending on whether the right or left side of the inequality (1.7) is violated for stresses satisfying (1.9).

For definiteness, we take (1.11) with the upper signs. Then, substituting (1.3) and (1.11) with account for (1.2) and (1.10) into (1.1), we obtain the following relations between the forces and deformations:

$$\mathbf{t}' = \|a'_{km}\|\,\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \|b'_{km}\|\,\mathbf{t}, \qquad \|b'_{km}\| = \|a'_{km}\|^{-1}, \;\; \mathbf{t}' = \|t_1', t_2', t_3'\|' \qquad (k, m = 1, 2, 3) \tag{1.12}$$

$$t_i' = t_i - \beta_n l_{in}^2, \qquad t_3' = t_3 - \beta_n l_{1n} l_{2n}, \qquad \beta_n = \omega_n \sigma_n^+ [1 - E_{tn}^+ / E_n]$$

In (1.12) the coefficients of the matrix $\|a'_{mk}\|$ have the same form as those of the matrix $\|a_{km}\|$ from (1.6), if in the latter E_q is replaced by $E_{tq}^{\pm}$ for $n = q$, where q is the number of the filament family which has changed into the plastic state.

The relations (1.12) describe the elastoplastic behavior of the reinforced layer in the case in which the plasticity condition for the layer has the form (1.9). These relations will be valid until some other of the inequalities (1.7) or (1.8) is violated, provided that b_{ik} and t_k in the latter are replaced respectively by b_{ik}', t_k' ($i = 1, 2$; $k = 1, 2, 3$). In the case of violation of inequalities of the type (1.7), the subsequent

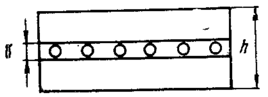

Fig. 1

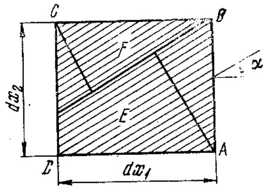

Fig. 2

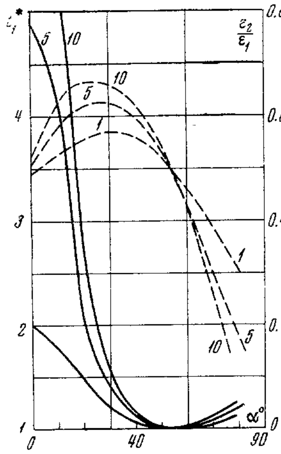

Fig. 3

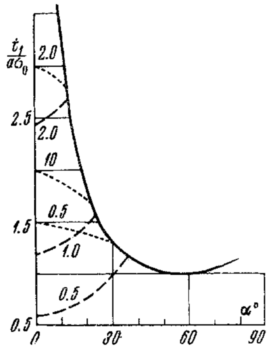

Fig. 4

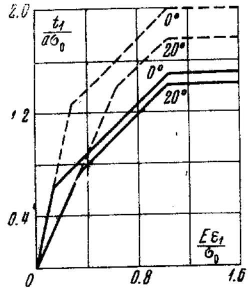

Fig. 5

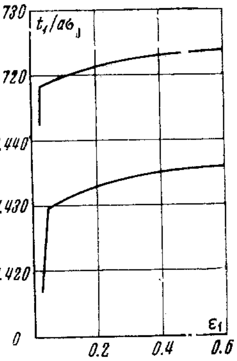

Fig. 6

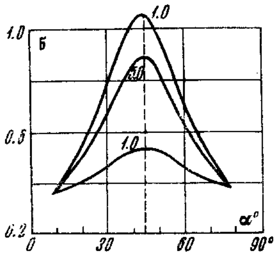

Fig. 7

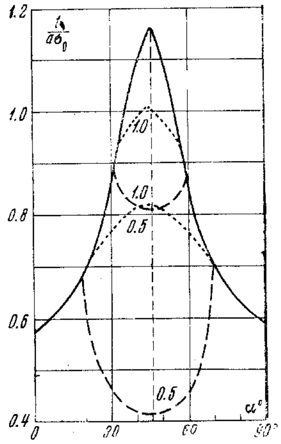

Fig. 8

modifications of the equations can be accomplished just as in obtaining (1.12), and as a result we shall have linear equations similar to (1.12) every time. If the material of the reinforcing elements is ideally elastoplastic, the corresponding tangent moduli $E_{tn}^{\pm}$ should be equated to zero in (1.12) and in the similar equations. In this case, in contrast with the isotropic or "physically anisotropic" ideally plastic layer, the reinforced layer with ideal elastoplastic reinforcing elements provides one-to-one dependence between the stresses and strains for inelastic deformations as well.

If the inequality (1.8) is violated first, while (1.7) remains valid, plastic deformations begin in the filler layer and are accompanied by elastic deformations in the reinforcing elements. Then we must take

$$E_{cn}^{\pm} = E_n, \quad E_c = \Phi(\varepsilon), \quad \varepsilon = {}^2/_3 \sqrt{3}[\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2^2 + {}^1/_4\varepsilon_3^2]^{1/2} \tag{1.13}$$

Here ε is the strain intensity in the filler layer, and Φ is a function determined from the plastic part of the stress-strain diagram.

Substituting for the secant moduli (1.13) the expressions (1.3) and (1.2) into (1.1), we obtain

$$\begin{aligned} t_1 &= a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + a_{13}\varepsilon_3 + {}^4/_3[\Phi(\varepsilon) - E](\varepsilon_1 + {}^1/_2\varepsilon_2) \\ t_2 &= a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + a_{23}\varepsilon_3 + {}^4/_3[\Phi(\varepsilon) - E](\varepsilon_2 + {}^1/_2\varepsilon_1) \\ t_3 &= a_{13}\varepsilon_1 + a_{23}\varepsilon_2 + a_{33}\varepsilon_3 + {}^1/_3[\Phi(\varepsilon) - E]\varepsilon_3 \end{aligned} \tag{1.14}$$

In contrast with (1.12), these equations are essentially nonlinear even for linear hardening of the filler material. If the filler material is ideally plastic with the yield point σ_0, in (1.14) we must take $\Phi(\varepsilon) = \sigma_0/\varepsilon$. In this case (1.14) define a one-to-one relationship between t_k and ε_k, provided $a_{km} \neq 0$ (k, m = 1, 2, 3).

If the inequality (1.7) is violated first, so that plastic deformations appear in the reinforcing elements which form the angle α_n with the direction 1, and the deformation of the reinforced material takes place in accordance with the relations (1.12), and then the inequality (1.8) is violated for some values t_k, subsequently the deformation law will have the form (1.14) if in place of t_k, a_{km} we substitute t_k' and a_{km}', respectively.

Relations of the type (1.12) and (1.14) not only describe the nature of the elastoplastic layer deformation, but together with (1.2) and (1.3) they also determine the effectiveness of the operation of all the elements of the composite. Depending on which elements of the composite deform plastically, these relations have a linear or nonlinear nature; therefore, in many cases some indirect information on the effectiveness of a given type of reinforcement can be obtained directly from the experimental deformation diagrams of the materials or structures.

2. As an example of the use of the relations obtained, we shall examine the problem of stretching by the force t_1 of a layer reinforced by unidirectional filaments which forms the angle $\alpha_1 = \alpha$ with the loading direction. For simplicity we shall assume that the material of all elements of the composite is ideally elastoplastic. Then in the case in question we must take

$$t_2 = t_3 = 0, \quad N = 1, \quad E_{t_1}^{\pm} = 0, \quad \Phi(\varepsilon) = \sigma_0 / \varepsilon$$

Therefore, we obtain in the elastic region

$$\frac{t_1}{aE\varepsilon_1} = \delta, \quad \frac{\sigma_1}{aE\varepsilon_1} = \frac{E_1}{2Ea}\delta_1, \quad \frac{\varepsilon_2}{\varepsilon_1} = -\frac{\Delta_1}{\Delta}, \quad \frac{\varepsilon_3}{\varepsilon_1} = -\frac{\Delta_2}{\Delta} \tag{2.1}$$

$$\frac{\sigma_{11}^{\circ}}{E\varepsilon_1} = \frac{4}{3}\left(1 - \frac{\Delta_1}{2\Delta}\right), \quad \frac{\sigma_{22}^{\circ}}{E\varepsilon_1} = \frac{4}{3}\left(\frac{1}{2} - \frac{\Delta_1}{\Delta}\right), \quad \frac{\sigma_{12}^{\circ}}{E\varepsilon_1} = -\frac{\Delta_2}{3\Delta} \tag{2.2}$$

$$\delta = \frac{a_{11}\Delta - a_{12}\Delta_1 - a_{13}\Delta_2}{aE\Delta}, \quad \delta_1 = \frac{2\Delta\cos^2\alpha - \Delta_2\sin 2\alpha - 2\Delta_1\sin^2\alpha}{\Delta}$$

$$\Delta = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a_{12} & a_{23} \\ a_{13} & a_{33} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{22} & a_{12} \\ a_{23} & a_{13} \end{vmatrix} \tag{2.3}$$

Figure 3 shows as a function of α the quantities $t_1^* = t_1/aE\varepsilon_1$ and $\varepsilon_2/\varepsilon_1$, calculated using (2.1) for $\omega_1 E_1/aE = 1, 5, 10$.

In this case the plasticity condition for the reinforcing elements has the form

$$\frac{t_1}{a\sigma_0} = \pm\frac{2\delta E\sigma_1^{\pm}}{\delta_1 E_1\sigma_0} \tag{2.4}$$

The upper signs correspond to $\varepsilon_{(1)} > 0$ ($\varepsilon_{(1)}$ is the deformation of the reinforcing elements calculated using (1.2) with account for (2.1)), the lower signs are for $\varepsilon_{(1)} < 0$. We further assume for definiteness that $\sigma_1^+ = \sigma_1^-$. Then it is not difficult to see that we need take only the upper signs in (2.4). The dashed curves in Fig. 4 show $t_1/a\sigma_0$ as a function of α, calculated for $\omega_1\sigma_1^+/a\sigma_0 = 0.5$, 1.0, 2.0, and $\omega_1 E_1/Ea = 5$.

For load values located between the dashed and dotted curves in Fig. 4, the stresses in the reinforcing elements remain constant and equal to σ_1^+. The relationships between the force and deformations have the form

$$aE\varepsilon_1 = t_1 - \omega_1\sigma_1^+(\cos^2\alpha - {}^1/_2\ \sin^2\alpha),\quad aE\varepsilon_2 = -{}^1/_2 t_1 - \omega_1\sigma_1^+(\sin^2\alpha - {}^1/_2\cos^2\alpha)$$
$$aE\varepsilon_3 = {}^3/_2\ \omega_1\sigma_1^+\ \sin 2\alpha \tag{2.5}$$

Therefore, the shear deformation is independent of the load while the "Poisson coefficient," conversely, depends on the load.

The relations (2.5) are valid up to load values at which plastic deformations arise in the binder. The corresponding load values are found from the equation

$$(t_1 - \omega_1\sigma_1^+\cos^2\alpha)^2 + (t_1 - \omega_1\sigma_1^+\cos^2\alpha)\omega_1\sigma_1^+\sin^2\alpha + (\omega_1\sigma_1^+)^2\sin^4\alpha + {}^1/_4(\omega_1\sigma_1^+)^2\sin^2 2\alpha = (a\sigma_0)^2 \tag{2.6}$$

The dependence of $t_1/\sigma_0 a$ on α, calculated with the aid of this equation for $\omega_1\sigma_1^+/a\sigma_0 = 0.5$, 1.0, 2.0, is shown by the dotted curves in Fig. 4. The reinforced material with ideally plastic composite elements cannot withstand large loads.

Formulas (2.1) and (2.5) define the stress-strain diagram of the material in those cases in which plastic deformations first develop in the reinforcing elements. The corresponding diagrams for $\omega_1 E_1/aE = 5$, $\omega_1\sigma_1^+/a\sigma_0 = 0.5$ (solid) and 1.0 (dashed), and $\alpha = 0, 20°$ are shown in Fig. 5. The horizontal segments correspond to the limit loads for the layer.

Also possible is the case in which plastic deformations appear first in the binder, while the reinforcing elements remain elastic. Then in (1.10) we must replace the inequality symbol by an equality symbol, and after substitution herein of (2.2), we obtain the relation

$$2t_1 / a\sigma_0 = \sqrt{3}\delta\Delta(\Delta^2 - \Delta\Delta_1 + \Delta_1^2)^{-1/2} \tag{2.7}$$

This relation is shown by the solid curve in Fig. 4 for $\omega_1 E_1/aE = 5$.

For loads exceeding the values (2.7), using for simplicity the ideally plastic material assumption, we have the following relationships between the force t_1 and the deformations ε_1, ε_2, ε_3:

$$2a(2\varepsilon_1 + \varepsilon_2)\sigma_0 + 3\omega_1 E_1\varepsilon\varepsilon_1\ \cos^2\alpha = 3t_1\varepsilon,\quad 2a(2\varepsilon_2 + \varepsilon_1)\sigma_0 + 3\omega_1 E_1\varepsilon\varepsilon_1\ \sin^2\alpha = 0,$$
$$2a\varepsilon_3\sigma_0 + 3\omega_1 E_1\varepsilon\varepsilon_1\ \sin 2\alpha = 0 \tag{2.8}$$

By simple transformations we obtain from these equations the following relations:

$$6\,\mathrm{tg}\,\alpha(1 - Bt_1/\sigma_0 a) + (\mathrm{tg}^2\alpha - 2)\quad \varepsilon_3/\varepsilon_1 = 0$$
$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{1}{4}\left(\frac{\varepsilon_3}{\varepsilon_1}\mathrm{tg}\,\alpha - 2\right),\quad \varepsilon_1 = \frac{1}{A}\left[2\left(\frac{Bt_1}{a\sigma_0} - 1\right) - \frac{\varepsilon_3}{\varepsilon_1}\mathrm{tg}\,\alpha\right]$$
$$A = \frac{1}{4(1+\mathrm{tg}^2\alpha)}\left[\frac{\varepsilon_3}{\varepsilon_1}\mathrm{tg}\,\alpha(\mathrm{tg}^2\alpha + 4) - 2(\mathrm{tg}^2\alpha - 2)\right],\quad B = \frac{1}{2\sqrt{3}}\left[12 + \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^2(\mathrm{tg}^2\alpha + 4)\right]^{1/2} \tag{2.9}$$

The stresses in the reinforcing elements are

$$\omega_1\sigma_1 / a\sigma_0 = A / B \tag{2.10}$$

Formulas (2.9) define a unique relationship between the force t_1 and the deformations ε_1, ε_2, ε_3, in spite of the fact that the binder is ideally plastic. We also emphasize that in contrast with (2.5) this relationship is nonlinear. Figure 6 shows the stress-strain diagrams calculated using (2.1) and (2.9) for

$$E/\sigma_0 = 40,\quad \omega_1 E_1 / aE = 1.5$$

The relations (2.9) will be valid up to those values of t_1 for which the stresses in the reinforcing elements reach the elastic limit. This load value will be limiting for the given material. A horizontal segment appears on the stress-strain diagram at this load. In Fig. 6 the straight lines correspond to the solution (2.1) and the curved segments correspond to the solution (2.9). The transition points are defined using (2.7).

In concluding this section, we note that (2.1) and (2.5) can also be used to describe the stress-strain diagram of a material reinforced by ideally brittle filaments. In this case (2.4) defines the loads at which filament breaking takes place. If the yield condition for the filler is not violated at these load values, the subsequent behavior of the material is characterized by an instantaneous jump on the diagram, described by (2.5) for $\sigma_1^+=0$.

3. As the second example, we shall analyze the shear of a unidirectionally reinforced material. In this case

$$t_1 = t_2 = 0, \quad N = 1, \quad \alpha_1 = \alpha, \quad E_{t1}^{\pm} = 0, \quad \Phi(\varepsilon) = \sigma_0 / \varepsilon$$

In the elastic region we have

$$\frac{t_3}{aE\varepsilon_3} = \delta, \quad \sigma_1 = \frac{E_1\delta_1}{aE\delta} t_3, \quad \frac{\sigma_{11}^0}{E} = -\frac{2t_3(2\Delta_1+\Delta_2)}{3\delta\Delta Ea}$$
$$\frac{\sigma_{22}^0}{E} = -\frac{2(2\Delta_2+\Delta_1)t_3}{3E\delta\Delta a}, \quad \frac{\sigma_{12}^0}{E} = \frac{t_3}{3\delta Ea} \tag{3.1}$$

Here

$$\delta = \frac{a_{33}\Delta - a_{13}\Delta_1 + a_{23}\Delta_2}{aE\Delta}, \quad \delta_1 = \frac{(\Delta \sin 2\alpha - 2\Delta_1 \cos^2\alpha - 2\Delta_2 \sin^2\alpha)}{2\Delta}$$
$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Figure 7 shows curves of the shear modulus versus α, calculated using (3.1) for $\omega_1 E_1/aE = 1, 5, 10$.

In this case the plasticity condition for the reinforcing elements has the form

$$t_3 / a\ \sigma_0 = E\delta\sigma_1^+ / E_1\delta_1\sigma_0 \tag{3.2}$$

Curves of this relation for $\omega_1 E_1/aE = 5$ and $\omega_1\sigma_1^+/a\sigma_0 = 0.5$, 1.0 are shown dashed in Fig. 8. If the loads exceed the values (3.2), the elastoplastic behavior is defined by the relations

$$aE\varepsilon_1 = -\omega_1\sigma_1^+(\cos^2\alpha - {}^1/_2 \sin^2\alpha), \quad aE\varepsilon_2 = -\omega_1\sigma_1^+(\sin^2\alpha - {}^1/_2\cos^2\alpha), \quad aE\varepsilon_3 = 3[t_3 - {}^1/_2\,\omega_1\sigma_1^+ \sin 2\alpha],$$
$$a\mathring{\sigma}_{11} = -\omega_1\sigma_1^+\cos^2\alpha, \quad a\mathring{\sigma}_{22} = -\omega_1\sigma_1^+\sin^2\alpha, \quad a\mathring{\sigma}_{12} = t_3 - {}^1/_2\,\omega_1\sigma_1^+\sin 2\alpha \tag{3.3}$$

Relations (3.3) are valid as long as the stresses σ_{11}^0, σ_{22}^0, σ_{12}^0 do not violate the inequalities (1.10). The corresponding limit load is defined by the equality

$$\frac{t_3}{a\sigma_0} = \frac{1}{2}\frac{\omega_1\sigma_1^+}{a\sigma_0}\sin 2\alpha + \frac{\sqrt{3}}{3}\left[1 - \left(\frac{\omega_1\sigma_1^+}{a\sigma_0}\right)^2(\cos^4\alpha - \sin^2\alpha\cos^2\alpha + \sin^4\alpha\right]^{1/2}$$

Curves of this load as a function of α for $\omega_1\sigma_1^+/a\sigma_0 = 0.5$, 1.0 are shown dotted in Fig. 8. The solid curves in Fig. 8 show the loads versus α for which plastic deformations arise first in the binder. We obtain the corresponding equations for these curves if in (1.8) we replace the inequality symbol by an equality symbol and then substitute therein the expressions (3.1) for σ_{11}^0, σ_{22}^0, σ_{12}^0.

We obtain the equations for the elastoplastic behavior of the reinforced layer for plastic deformations of the binder and elastic deformations of the reinforcing elements in this case from (1.17) for $t_1 = t_2 = 0$ and $\Phi(\varepsilon) = \sigma_0 / \varepsilon$.

Similar solutions can be obtained for any other types of loading or reinforcement of the layer.

In conclusion, we note that the ideally plastic material model used in the calculations was employed for definiteness and to emphasize certain peculiarities of the relationships between the stresses and deformations during elastoplastic deformation of reinforced materials. If necessary, the corresponding calculations can be made without difficulty for any concrete hardening law.

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